

## Section 9.4 part 1

group action

9.4

## Proof of Sylow Theorems

Recall

Action of  $G$  (group) on  $S$  (set). (Left)

- homomorphism  $G \rightarrow A(S)$

$A(S) = \{f: S \rightarrow S \mid f \text{ is a bijection}\}$   
group operation - composition  
of functions

Notation:

$$g \mapsto f_g$$

We write

$$g \cdot x = f_g(x) \quad g \in G, x \in S$$

$$\begin{array}{ccc} f_g: S & \rightarrow & S \\ x & \mapsto & f_g(x) \end{array}$$

We have  $(g_1 \cdot g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$

$$\left\{ \begin{array}{l} f_{g_1 \cdot g_2}(x) = f_{g_1}(f_{g_2}(x)) \end{array} \right.$$

- the map  $G \rightarrow A(S)$  is  
a group homomorphism

Orbit of  $x$ :  $\text{Orb}(x) = \{g \cdot x \mid g \in G\} \subseteq S$

The relation on  $S$  define by  $x \sim y$  iff  $x$  and  $y$  are on the same orbit  
is an equivalence relation  
that is  $y = g \cdot x$  for some  $g \in G$

$S = \bigcup_{x \in S} \text{Orb}(x)$  - partition of  $S$   
into non-overlapping

$$g'y = x$$

subsets - equivalence classes with respect to  
the relation  $\sim$  on  $S$

For an element  $x \in S$  we define the stabilizer of  $x$ :

$$St(x) = \{g \in G \mid g \cdot x = x\} \subseteq G$$

For any  $x \in S$ ,  $St(x)$  is a subgroup of  $G$

$G = \bigcup_{a \in G} a St(x)$  - the group is partitioned into left cosets

Relation between  $Orb(x) \subseteq S$  and

$$St(x) \subseteq G$$

$Orb(x) = \{g \cdot x \mid g \in S\} = \{a s \cdot x \mid s \in St(x)\}$ ,  $a$  - a representative of a left coset

$$a s \cdot x = a(s \cdot x) = a \cdot x$$

Choose one representative  $a \in G$  for every coset in  $G = \bigcup_a a St(x)$

$$= \{a \cdot x \mid a - \text{the representative}\}$$

There are exactly as many elements in  $Orb(x)$  as there are cosets

Pf (using Th 7.11)

$$g_1, g_2 \in St(x)$$

$$(g_1 g_2) \cdot x = g_1(g_2 \cdot x)$$

$$= g_1 \cdot x = x \quad g_1, g_2 \in St(x)$$

$$g \in St(x)$$

$$\underline{g^{-1} \cdot x} = \underline{g^{-1} \cdot (g \cdot x)}$$

$$= (\underline{g^{-1} g}) \cdot x = e \cdot x = x$$

$$\underline{g^{-1}} \in St(x)$$

$[G : St(x)]$

Prop Let  $x \in S$ .

Assume that either the set  $Orb(x) \subseteq S$  or the index  $[G : St(x)]$  is finite. Then so is another one, and we have that

$$\underline{|Orb(x)| = [G : St(x)]}$$

$$|Orb(x)| = |G| / St(x)$$

In particular,  $|Orb(x)| / |G|$

When  $G$  is finite, we have

$$[G : St(x)] = |G| / St(x)$$